Supplement 7--A guide to probabilities in Chapter 4

For all the examples which follow, assume a bag filled with 100 marbles of the following colors and styles:

<table>
<thead>
<tr>
<th></th>
<th>Cat’s Eye</th>
<th>Moonie</th>
<th>Solid</th>
<th>Σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blue</td>
<td>13</td>
<td>6</td>
<td>13</td>
<td>32</td>
</tr>
<tr>
<td>Red</td>
<td>7</td>
<td>4</td>
<td>12</td>
<td>23</td>
</tr>
<tr>
<td>Yellow</td>
<td>12</td>
<td>4</td>
<td>9</td>
<td>25</td>
</tr>
<tr>
<td>Green</td>
<td>4</td>
<td>2</td>
<td>14</td>
<td>20</td>
</tr>
<tr>
<td>Σ</td>
<td>36</td>
<td>16</td>
<td>48</td>
<td>100</td>
</tr>
</tbody>
</table>

Note that the color of a marble (row variable) and its style (column variable) are mutually exclusive.

**Simple probability**

To calculate the probability of a single event occurring on a single trial, we use the formula

\[
P(\text{event}) = \frac{\# \text{ successful possible outcomes}}{\# \text{ total possible outcomes}}
\]

So, the probability that Sally will select a blue marble from the bag on a single trial is

\[
P(\text{blue}) = \frac{\# \text{ blue marbles}}{\# \text{ marbles}} = \frac{32}{100} = .32
\]

(NOTE: You’ll find that many of the more complicated probability formulas are just variations on this one. So, learn it well and keep it in mind as we work through the rest.)

**Addition and multiplication rules of probability**

The key to determining whether a problem is an addition rule problem or multiplication rule problem is to determine the number of outcomes being considered and the number of trials being conducted.

- **One trial only**, find probability of a subset of multiple possible outcomes ⇒ Addition Rule
- **Multiple trials**, find probability of only one outcome on each trial ⇒ Multiplication Rule

**Addition Rule**

There are two of these:

- non-mutually exclusive outcomes ⇒ General Addition Rule
- mutually exclusive outcomes ⇒ (Special) Addition Rule for Mutually Exclusive Events

For simplicity, I’ll refer to them as the General and Special Rules of Addition.

**Examples:**

1. Sally replaces her marble, and then Bobby takes one marble out of the bag. What is the probability that Bobby’s marble is either yellow or a moonie?
   → A single marble can be both yellow and a moonie, since these involve two different variables, so
   **One trial only & multiple, non-mutually exclusive outcomes** ⇒ General Addition Rule
   
   Letting A be a yellow marble and B be a moonie, we have
   
   \[P(A \cup B) = P(A) + P(B) - P(A \cap B) = .25 + .16 - .04 = .37\]

2. Say that Sally takes one marble out of the bag. What is the probability that it is either blue or red?
   → Since a marble cannot be both blue and red, that is, “blue” and “red” are values of the same variable,
   **One trial only & multiple, mutually exclusive outcomes** ⇒ Special Addition Rule
   
   Letting A be a blue marble and B be a red marble, we have
   
   \[P(A \cup B) = P(A) + P(B) = .32 + .23 = .55\]
Multiplication Rule

There are two of these, too, also called the General and Special Rules. Spot the difference:

<table>
<thead>
<tr>
<th>1</th>
<th>independent events</th>
<th>Probability of success changes from one trial to the next</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>dependent events</td>
<td>General Multiplication Rule</td>
</tr>
<tr>
<td>2</td>
<td>independent events</td>
<td>Probability of success remains the same from trial to trial</td>
</tr>
<tr>
<td></td>
<td>⇒ sampling without replacement</td>
<td></td>
</tr>
<tr>
<td></td>
<td>⇒ General Multiplication Rule</td>
<td></td>
</tr>
<tr>
<td></td>
<td>⇒ sampling with replacement</td>
<td></td>
</tr>
<tr>
<td></td>
<td>⇒ (Special) Multiplication Rule for Independent Events</td>
<td></td>
</tr>
</tbody>
</table>

Let's review the concept of statistical independence: Two events are said to be statistically independent if the occurrence of one event has no influence on the likelihood of occurrence of the other.

Why is sampling with replacement equivalent to statistical independence? Back to our bag of marbles. Statistical independence requires that, no matter how many blue marbles we select, the probability of selecting a blue marble (or any color marble, for that matter) changes. There are two ways you could do this: One, to have an infinitely large bag of marbles, since taking one blue marble out of the bag would reduce neither the number of marbles in the bag, nor the number of blue marbles (maybe this isn’t intuitive, but remember that $\infty - 1 = \infty$). A second, more practical way, would be to replace each marble before making another selection, keeping the proportions, and hence the probabilities, of blue and red marbles identical from one trial to the next. That is what is meant by sampling with replacement. You just keep “playing the same game” over and over.

Now, some examples:

1. Bobby starts with a full bag of marbles. He takes one marble out of the bag. Without replacing it, he takes a second. What is the probability that both Bobby’s marbles are red?
   
   Multiple trials & one outcome & dependent events (sampling without replacement) $\Rightarrow$ general rule of multiplication

   Letting A be “red marble, first selection” and B be, “red marble, second selection,” we have
   
   \[
   P(A \cap B) = P(A) \times P(B|A) = .23 \times .2222 = .0511
   \]

   (Note: $P(B|A) = \#$ red marbles remaining in bag $\div \#$ all remaining marbles $= 22 \div 99 = .2222$)

2. Let’s start with the full bag of marbles again. Say that Sally takes one marble out of the bag, then replaces it, and takes out another. What is the probability that both are blue?

   Multiple trials & one outcome & independent events (sampling with replacement) $\Rightarrow$ special rule of multiplication

   Letting A be “blue marble, first selection” and B be, “blue marble, second selection,” we have

   \[
   P(A \cap B) = P(A) \times P(B) = .32 \times .32 = .1024
   \]

3. OK, now Sally puts the second marble back, then repeats the procedure. What is the probability that, this time, she gets a blue marble on the first trial and a red marble on the second trial?

   Letting A be “blue marble, first selection” and B be, “red marble, second selection,” we have

   \[
   P(A \cap B) = P(A) \times P(B) = .32 \times .23 = .0736
   \]

4. Of course, Sally could have gotten a red marble first and a blue one second. Either way, she would have selected one blue and one red marble, total. What’s the probability of her selecting one blue and one red marble, not caring about the order of selection? (Oops, sorry; this requires us to calculate either a binomial probability or a Poisson probability. We’ll have to wait til Chapter 5.)

Complement rule

Sometimes it’s easier to calculate the probability of what you don’t want and then work backward to find the probability you’re seeking. That’s the idea of the complement rule:

\[
P(A) = 1 - P(A')
\]

where $A'$ (read as “A prime,” or “A complement,” or simply “not A”) is the complement of A. So, referring back to our very first example problem, the probability that Sally will select a blue marble from the bag on a single trial is given by

\[
P(\text{Blue}) = 1 - P(\text{Blue}') = 1 - P(\text{Red U Yellow U Green}) = 1 - [P(\text{Red}) + P(\text{Yellow}) + P(\text{Green})] = 1 - [.23 + .25 + .2] = .32
\]

(I use the Special Addition Rule to help with this.)
Decision chart for addition and multiplication rule probabilities

Here's a handy decision chart to help you determine the type of problem you are dealing with.